Design of Unimodular Sequences using Generalized Receivers

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***Abstract*— This paper reviews recent advances in designing unimodular sequences with good auto/cross correlation properties along with a new approach that emphasizes on independent receiver design. The general problem is to design single or multiple sequences with constant modulus in the time-domain such that their respective matched filter outputs ideally resemble delta functions and the cross-matched filter outputs are zeros. In this context CAN (cyclic-algorithm new) and WeCAN (Weighted CAN) have been proposed for designing such sequences with good auto-cross correlation properties. In this paper, the equivalence of the CAN algorithms and the classic Gerchberg-Saxton (GS) algorithm involving the sequential magnitude substitution operations in the time and frequency domain is demonstrated.**

The design of unimodular sequences is further generalized here by considering the receiver design to be more general than the respective matched input sequences. The receiver design is carried out by taking care of the desired output requirements and the freedom present at the input can be used to further minimize the output side-lobe level.

I. INTRODUCTION

In a series of excellent papers, Stoica et. al. has recently addressed the problem of designing unimodular sequences with good auto/cross correlation properties [1, 2, 3]. The problem is to design single or multiple sequences with constant modulus in the time-domain such that their respective matched filter outputs ideally resemble delta functions while the cross-matched filter outputs are zeros. Since the matched filter outputs represent auto and cross-correlations of the original sequences, the problem is to design constant modulus time-series whose auto correlations approximate the delta function, and the cross correlations approximate the all-zero function. In this context CAN (cyclic-algorithm new) and WeCAN (Weighted CAN) have been proposed for designing such sequences with good auto-cross correlation properties [1, 3]. In this correspondence, the equivalence of the CAN algorithm and the classic Gerchberg-Saxton algorithm is demonstrated [4]. In the multichannel case, the CAN/WeCAN algorithms coincide with the Gerchberg-Saxton algorithm in the time domain, and they generalize into the frequency domain in an interesting manner.

The design of unimodular sequences is further generalized here by considering the receiver design to be something other than their respective matched filters. The receivers need not have the unimodular property, and instead they can emphasize on the desired impulse-like properties. This approach gives rise to more efficient receiver design that realize the desired output sequences and the freedom present at the input can be used to accommodate additional constraints.

II. GERCHBERG-SAXTON ALGORITHM

The Gerchberg-Saxton algorithm considers the problem of reconstructing a time function with known partial magnitude information both in the time and frequency domain. Thus let

$$f(t) = a(t)\, e^{j\theta(t)} \leftrightarrow F(\omega) = M(\omega)\, e^{j\psi(\omega)} \tag{1}$$

represent a Fourier transform pair. Given the information

$$a(t) \geq 0, \quad 0 < t < T, \quad \text{and} \quad \mathrm{M}(\omega) \geq 0, \ |\omega| < B_o, \tag{2}$$

both in time and frequency domains, the problem is to reconstruct the entire $f(t)$ in some optimal fashion. It is well known that the set of all signals that have a prescribed value in the interval $(0, T)$ forms a closed convex set C. Convex sets have the property that for any point $f(t)$ outside C, there exists a *unique* nearest neighbor $Pf(t)$ in C such that [5].

$$\| f(t) - Pf(t) \| \leq \| f(t) - g(t) \|, \ \text{for all } g(t) \in C. \tag{3}$$

Although the set of all signals C_M with the given magnitude transform $M(\omega)$ over a prescribed bandwidth Ω *do not* form a closed convex set [5], nevertheless, it is possible to assign to every arbitrary signal $f(t)$ that is outside this set a "nearest neighbor" signal $P_M f(t)$ that belongs to C_M such that there exists no other signal $g(t) \in C_M$ for which

$$\| f(t) - g(t) \| < \| f(t) - P_M f(t) \| \tag{4}$$

is satisfied. In the case of closed convex sets the above operator is the projection operator and the "nearest neighbor" is unique [5]. In the case of the above magnitude substitution operator, the uniqueness property is not preserved for the "nearest neighbor" and in general only the inequality constraint

$$\| f(t) - P_M f(t) \| \leq \| f(t) - g(t) \|, \ \text{for all } g(t) \in C_M \tag{5}$$

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14. ABSTRACT
This paper reviews recent advances in designing unimodular sequences with good auto/cross correlation properties along with a new approach that emphasizes on independent receiver design. The general problem is to design single or multiple sequences with constant modulus in the timedomain such that their respective matched filter outputs ideally resemble delta functions and the cross-matched filter outputs are zeros. In this context CAN (cyclic-algorithm new) and WeCAN (Weighted CAN) have been proposed for designing such sequences with good auto-cross correlation properties. In this paper, the equivalence of the CAN algorithms and the classic Gerchberg-Saxton (GS) algorithm involving the sequential magnitude substitution operations in the time and frequency domain is demonstrated. The design of unimodular sequences is further generalized here by considering the receiver design to be more general than the respective matched input sequences. The receiver design is carried out by taking care of the desired output requirements and the freedom present at the input can be used to further minimize the output side-lobe level.

15. SUBJECT TERMS

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is satisfied. Given an arbitrary $f(t)$, to determine $P_M f(t)$, we need to generate $g(t) \in C_M$ such that $\|f(t) - g(t)\|^2$ is minimum over Ω. With $f(t) \leftrightarrow F(\omega) = |F(\omega)| e^{j\psi(\omega)}$,

$$P_M f(t) \leftrightarrow \begin{cases} M(\omega) e^{j\psi(\omega)}, & \omega \in \Omega \\ F(\omega), & \omega \in \Omega' \end{cases} \tag{6}$$

gives the magnitude substitution operator [4, 6, 7].

Similarly, any prescribed temporal magnitude function $a(t) > 0$ as in (2), or in particular, a constant envelope signal $f(t)$ can be expressed as $f(t) = Ae^{j\theta(t)}$ where A is a suitable positive constant that can be used to maintain a prescribed energy level for $f(t)$. It is easy to see that signals with the same constant envelope *do not* form a convex set; however their behavior is similar to the signal set with given Fourier transform magnitude. Hence if C_A denotes the set of functions $\{g(t)\}$ that maintain a constant envelope level A, although C_A is not a convex set, it is possible to define an operator P_A that assigns to every arbitrary function $f(t)$ a nearest neighbor $P_A f(t)$ that belongs to C_A as in (5). Thus with $f(t) = b(t) e^{j\theta(t)}$, we have

$$P_A f(t) = \begin{cases} Ae^{j\theta(t)}, & t \in (0, T) \\ f(t), & otherwise \end{cases} \tag{7}$$

where the interval $(0, T)$ represents the duration over which the constant envelope property is to be maintained. The error between $f_k(t)$ and its constant envelope version $P_A f_k(t)$ or the transform magnitude substituted version of $P_M f_k(t)$ reduces as $k \to \infty$ [6].

More interestingly, the iteration that combines both the Fourier transform substitution operation as well as the constant envelope substitution operation has the desired error reduction properties. Thus with

$$f_{k+1}(t) = P_A P_M f_k(t) = P_A g_k(t) \tag{8}$$

where we define $g_k(t) = P_M f_k(t)$. In that case the error reduction property [4]

$$d_{k+1} = \|f_{k+1} - g_{k+1}\| = \|f_{k+1} - P_M f_{k+1}\| \le \|f_k - P_M f_k\| = d_k \tag{9}$$

is maintained since (see Fig. 1)

$$\begin{aligned} d_{k+1} &= \|f_{k+1} - P_M f_{k+1}\| \le \|f_{k+1} - P_M f_k\| \\ &= \|P_A P_M f_k - P_M f_k\| \le \|f_k - P_M f_k\| = d_k \ . \end{aligned} \tag{10}$$

In (8), $f_k(t)$ represents the constant envelope signal, whereas $g_k(t)$ preserves the desired Fourier transform magnitude function. Furthermore, the above error reduction property is maintained also if the above iteration includes convex projection operators P such as in the iteration

$$f_{k+1}(t) = P P_A P_M f_k(t) \, . \tag{11}$$

Here, for example, P can represent the projection operator that truncates the constant modular signal $P_A P_M f_k(t)$ to the interval $(0, T)$ obtained by substituting it to zero outside that interval. The iterative algorithm (11) also satisfies the error reduction property as in (9) - (10) [7]. Equations (8) - (11) represent the classic Gerchberg-Saxton algorithm [4].

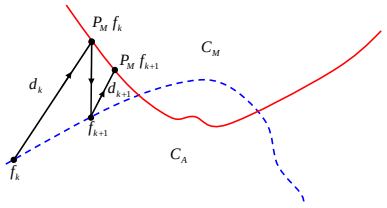


Fig. 1 Error reduction property for magnitude substitution operators.

Observe that the iteration in (8) is an excellent way to prescribe a given Fourier transform magnitude while maintaining a constant envelope in the time domain, and together they result in iterations that "get closer" in the sense of (9). From (8), $f_{k+1}(t)$ satisfies the constant modulus property while only approximating the given magnitude transform $M(\omega)$; whereas $g_k(t) = P_M f_{k+1}(t)$ possesses the given magnitude transform while only approximating the constant modulus property. Nevertheless, after a large number of iterations, as (9) shows, the difference between these signals gets smaller. Notice that since (9) - (10) do not guarantee convergence properties for $f_k(t)$, and in general there is no unique solution. The final solution is sensitive to the starting point $f_o(t)$, and a careful selection of the initial solution based on other factors is essential [4, 7, 8, 9].

In particular, the discrete version of the Gerchberg-Saxton algorithm described above can be summarized as follows [10]:

Let $\{f_i\}_{i=1}^N$ and $\{F_k\}_{k=1}^N$ form an N-point DFT pair at $\omega_k = 2\pi k/N$, $k = 1 \to N$, whose magnitude values $|f_i| = a_i$ and $|F_k| = A_k$ are specified both in the time and frequency domain. Here $a_i = a$ for example will refer to the unimodular case. Perform the N-point DFT of the given sequence and substitute the frequency magnitude sequence to A_k, then perform the inverse DFT and substitute the temporal magnitude values to a_i and repeat the procedure. For example, Wikipedia summarizes the above algorithm as [10]:

```
Gerchberg–Saxton Algorithm(Source, Target,
Retrieved_Phase)
 A = IFT(Target)
 while error criterion is not satified
   B = Amplitude(Source) * exp(i*Phase(A))
   C = FT(B)
   D = Amplitude(Target) * exp(i*Phase(C))
   A = IFT(D)
 end while
 Retrieved_Phase = Phase(A)
   end Gerchberg–Saxton Algorithm
```

(12)

The error reduction property in (9) is guaranteed in this case also. Interestingly for the N-point discrete case, the above algorithm implicitly contains the temporal limiting projection

operator P described in (11) and hence it represents the iteration in (11).

III. Unimodular Sequences uisng CAN And WeCAN

In the single channel case, as Fig. 2 shows, the problem is to design a sequence $\{x_i\}_{i=1}^{N}$ such that $|x_i|=1$ for $i=1\to N$, and their matched filter outputs approximate an ideal delta function for pulse compression and sidelobe suppression. Since the matched filter outputs $\{r_k\}_{k=-(N-1)}^{N-1}$ are the autocorrelations of the input sequence, we have

$$r_k = \sum_{i=k+1}^{N} x_i\, x_{i-k}^{*} = r_{-k}^{*}, \quad k=0,\pm 1,\pm(N-1)\ . \tag{13}$$

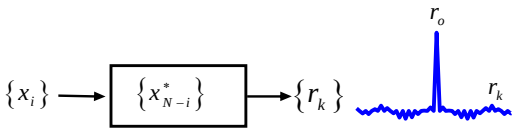


Fig. 2 Matched filter.

Hence for pulse compression and side lobe suppression, ideally we must have a delta function at the output, *i.e.*, $r_o=1$, $r_k=0,\ k\geq 1$ or in this context, minimization of the total sidelobe error [3]

$$\varepsilon=(r_o-1)^2+2\sum_{k=1}^{N-1}|r_k|^2 \tag{14}$$

has been suggested as a suitable criterion. The minimization of the above error is closely related to the minimization of the integrated sidelobe level (*ISL*)

$$ISL=\sum_{k=1}^{N-1}|r_k|^2, \tag{15}$$

or the maximization of the merit factor (*MF*) $MF=|r_o|^2/2ISL$ proposed in the literature [11,12]. It is easy to relate the error in (14) or the *ISL* in (15) to the classic discrete-time Fourier transform of the sequence $\{x_i\}$ given by $X(\omega)=\sum_{i=1}^{N}x_i\,e^{-ji\omega}$. Using (13), the corresponding spectrum $S(\omega)$ is given by

$$S(\omega)=|X(\omega)|^2=\sum_{k=-(N-1)}^{N-1}r_k\,e^{-jk\omega} \tag{16}$$

and hence when the ideal conditions where r_k is delta function are substituted into (16) we obtain the new requirement

$$S(\omega)\equiv 1 \tag{17}$$

for all ω in the frequency domain. Eq. (17) represents the well known result that to achieve excellent pulse compression, the input sequence must exhibit flat spectrum. Since the requirement in (17) is impossible to maintain over all frequencies by nontrivial finite sequences, to generate unimodular finite length sequences that satisfy (17), the optimization problem $\Delta_o=\min_{|x_i|=1}\|S(\omega)-I\|^2$ is more meaningful. It also involves the difficult task of minimizing the error over *all* frequencies. However, the relaxed criterion

$$\Delta_1=\frac{1}{2N}\sum_{n=1}^{2N}\|S(\omega_n)-I\|^2,\quad \omega_n=\frac{2\pi n}{2N} \tag{18}$$

obtained by averaging the spectral error only at $2N$ equally spaced discrete points is more appealing since it is also equal to the total sidelobe error ε in (14). Using (16), we also have

$$\Delta_1=\frac{1}{2N}\sum_{n=1}^{2N}\left(|X(\omega_n)|^2-1\right)^2 \tag{19}$$

and clearly (19) is minimized by letting [1]

$$X(\omega_n)=e^{j\psi_n},\quad n=1\to 2N\ . \tag{20}$$

$X(\omega_n)$ represent the DFT coefficients at frequencies $\omega_n=2\pi n/2N$, and constraint (20) is implementation of the unity magnitude substitution requirement in the discrete frequency domain as in (12) . Using the $2N\times 2N$ DFT matrix $\mathbf{D}$ whose $(i,\ j)^{th}$ element is given by $D_{i,k}=\frac{1}{\sqrt{2N}}e^{j2\pi ik/2N}$, we can rewrite the $2N$ equations in (20) compactly as

$$\mathbf{D}^{*}\underline{X}=\left[e^{j\psi_1},\ e^{j\psi_2},\ \cdots e^{j\psi_{2N}}\right]^{T}\triangleq\underline{V},\ or\ \ \underline{X}=\mathbf{D}\underline{V} \tag{21}$$

where the $2N\times 1$ vector $\underline{X}$

$$\underline{X}=\left[x_1,\ x_2,\ \cdots x_N,\ 0\ \cdots\ 0\right]^{T},\quad |x_i|=1 \tag{22}$$

represents the extended data vector with $x_i,\ i=1\to N$ representing the unimodular entries. Equations (21) - (22) suggest that for the above minimization problem, the $2N$-point DFT of the constant modulus data vector $\underline{X}$ in (22) in the temporal domain should generate another constant modulus vector $\underline{V}$ in the frequency domain and the CAN/WeCAN algorithms perform these operations by employing the magnitude substitution operation both in the frequency domain and the time domain sequentially till the desired accuracy is achieved [1, 2, 3]. But this is the same as a Gerchberg-Saxton algorithm described in (8) - (12). The CAN algorithm in (21) - (22) resets the second half of the temporal coefficients to zero in (22) at every stage, and it corresponds to the projection operator P associated with the time-limiting operation in (11).

IV. Multi-Channel Case

The multichannel version of the CAN approach also uses the Gerchberg-Saxton algorithm in the time domain while generalizing it in frequency domain. The $N\times M$ matrix [2, 3]

$$\mathbf{X}\triangleq\left[\underline{x}_1,\ \underline{x}_2,\ \cdots\ \underline{x}_M\right]\triangleq\begin{bmatrix}\underline{y}_1^{T}\\ \underline{y}_2^{T}\\ \vdots\\ \underline{y}_N^{T}\end{bmatrix} \tag{23}$$

containing the M temporal sequences

$$\underline{x}_i = \left[x_{i,1},\ x_{i,2},\ \cdots x_{i,N} \right]^T, \qquad i = 1 \to M, \quad \left| x_{i,j} \right| = 1 \tag{24}$$

represent M orthonormal sequences each with N unimodular samples, where the n^{th} row

$$\underline{y}_n^T = [x_1(n),\ x_2(n),\ \cdots\ x_M(n)] \tag{25}$$

in (23) corresponds to the M channel outputs at the n^{th} time instant. Under stationary assumptions, cross-correlations of different channel waveforms give rise to the auto/cross-correlation coefficients

$$r_{i,j}(k) = \sum_{n=1}^{N-k} x_i(n) x_j^*(n-k)\ . \tag{26}$$

Using (25), the $M \times M$ auto/cross-correlation coefficient matrices generated using (26) can be compactly expressed also using the $M \times 1$ column vectors $\underline{y}_n$ as

$$\mathbf{R}_k = \sum_{n=1}^{N-k} \underline{y}_n\, \underline{y}_{n-k}^* = \left(r_{i\,j}(k) \right), \quad k = 0 \to N-1 \tag{27}$$

where $\mathbf{R}_k$ represents the $M \times M$ cross-covariance matrix corresponding to lag k. For orthonormal sequences we require

$$\mathbf{R}_o = \mathbf{I}, \quad \mathbf{R}_k = 0, \quad k \geq 1. \tag{28}$$

Following (14)-(15), if we define the total sidelobe error criterion

$$\varepsilon = \left\| \mathbf{R}_o - \mathbf{I} \right\|^2 + 2 \sum_{k=1}^{N-1} \left\| \mathbf{R}_k \right\|^2\ , \tag{29}$$

then, as before with $\underline{Z}(\omega) = \sum_{i=1}^{N} \underline{y}_i\, e^{-ji\omega}$, we have the power spectral density matrix

$$\mathbf{S}(\omega) \triangleq \underline{Z}(\omega)\, \underline{Z}^*(\omega) = \sum_{k=-(N-1)}^{N-1} \mathbf{R}_k\, e^{-jk\omega} \tag{30}$$

so that once again the desired error criterion in (28) can be expressed as $\mathbf{S}(\omega) \equiv \mathbf{I}$. As in (18), the relaxed criterion

$$\Delta_1 = \frac{1}{2N} \sum_{k=1}^{2N} \left\| \mathbf{S}(\omega_k) - \mathbf{I} \right\|^2 = \varepsilon\ , \quad \omega_k = \frac{2\pi k}{2N} \tag{31}$$

is more useful in this context since it relates the point power spectral error with the total sidelobe error in (29). Expanding (31) using (30) we also obtain [3]

$$\varepsilon = \frac{1}{2N} \sum_{k=1}^{2N} \left(\left\| \underline{Z}(\omega_k) \right\|^2 - 1 \right)^2 + M - 1 \tag{32}$$

so that minimization of ε is achieved by setting

$$\underline{Z}(\omega_k) = \underline{\alpha}_k, \quad \left\| \underline{\alpha}_k \right\|^2 = 1, \ k = 1 \to 2N \tag{33}$$

in (32). Notice that $\underline{\alpha}_k$ in (33) are the 2N-point DFT vectors, and their normalization requirement is a magnitude constraint in the frequency domain. Once again, Eq. (33) can be compactly expressed as

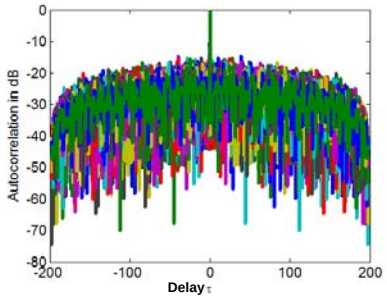
$$\begin{bmatrix} \underline{Z}^T(\omega_1) \\ \underline{Z}^T(\omega_2) \\ \vdots \\ \underline{Z}^T(\omega_{2N}) \end{bmatrix} = \mathbf{D} \begin{bmatrix} \mathbf{X} \\ \hline \mathbf{0} \end{bmatrix} = \begin{bmatrix} \underline{\alpha}_1^T \\ \underline{\alpha}_2^T \\ \vdots \\ \underline{\alpha}_{2N}^T \end{bmatrix} = \mathbf{V}, \ \left| x_{i\,j} \right| = 1, \left\| \underline{\alpha}_k \right\|^2 = 1 \tag{34}$$

where $\mathbf{X}$ is defined as in (23) with $\left| x_{i,j} \right| = 1$. Observe that the normalization of $\underline{\alpha}_k$ in the frequency domain in (33) - (34) is a generalization of the scalar version of the Gerchberg-Saxton algorithm since

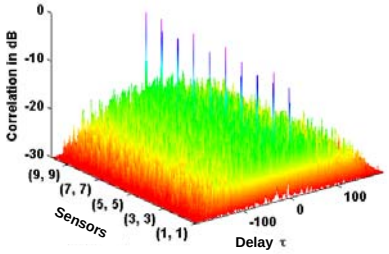
$$\left\| \underline{\alpha}_k \right\|^2 = \sum_{m=1}^{M} \left| Z_m(\omega_k) \right|^2 = M(\omega_k) \tag{35}$$

represents the sum of the magnitudes of the various channel components of the DFT coefficients. Following (33) - (34), the multichannel version of the CAN algorithm performs the magnitude substitution operation sequentially both in the frequency domain ($\left\| \underline{\alpha}_k \right\|^2 = 1$) and in the time domain ($\left| x_{i\,j} \right| = 1$) as in the Gerchberg-Saxton algorithm in (12).

Fig. 3 shows the matched filter receiver outputs of a ten channel unimodular sequence of length 200 generated using the CAN algorithm in (33) - (34). Both Fig. 3(a) and (b) contain ten auto-correlation sequences and 90 cross-correlation sequences. Notice that the auto/cross correlation terms are at about 20 dB below the peak term. The sidelobe levels here can be further improved by considering various weighting sequences as well [2, 3].



(a) Front view



(b) Side view

Fig. 3 A ten channel unimodular sequence design using the multichannel CAN algorithm in (33) - (34). Each figure contains ten autocorrelation sequences and 90 cross-correlation sequences.

V. General Unimodular Sequences

The design of unimodular sequences can be further generalized by considering the receiver design to be independent from the actual matched filter sequences. This approach is useful in many applications including radar, where the receiver does not have to be unimodular and can be something other than the matched filter sequence. In this context, the problem is to design M input unimodular sequences $\underline{x}_i$, $i=1\to M$, $|x_{i,j}|=1$ each of length N as in (23), and their corresponding receivers

$$\underline{h}_i=[h_{i,1},\ h_{i,2},\ \cdots h_{i,K}],\quad i=1\to M \tag{36}$$

such that the receiver outputs should satisfy the "desired" output sequence properties. Thus with

$$z_{i,m}(n)=\sum_{k=1}^{N}x_i(k)\,h_m(n-k+1),\ \ n=1\to N+K-1 \tag{37}$$

representing the output at the m^{th} receiver due to the i^{th} sequence at time instant n, then ideally, we require (Fig. 4)

$$z_{i,m}(n)=\begin{cases}\delta_{i,m}(n-n_o), & i=m\\ 0, & i\neq m\end{cases} \tag{38}$$

where n_o corresponds to the desired peak term at the output. In general, the receiver length K is a free parameter left to other design considerations.

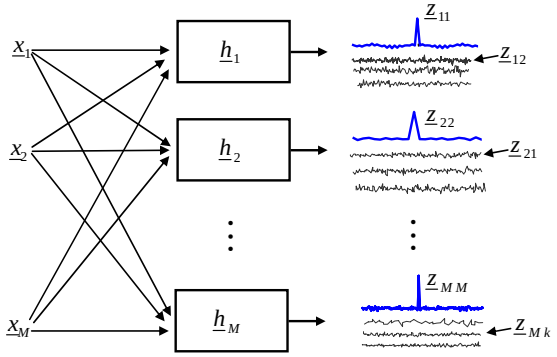


Fig. 4 Generalized unimodular sequences.

One approach in this context is to use a criterion similar to the merit factor (*MF*). Thus if we define

$$\eta_i=\frac{|z_{ii}(n_o)|^2}{\sum\limits_{n\neq n_o}|z_{ii}(n)|^2+\sum\limits_{j\neq i}\sum\limits_{n=1}^{N+K-1}|z_{ij}(n)|^2},\ \ i=1\to M \tag{39}$$

to represent the ratio of the receiver output peak value power to the total undesired sidelobe power at each receiver output, then the receiver design problem is to maximize η_i, $i=1\to M$. Observe that the criterion in (39) is well known and it is the same as the approach behind the classical matched filter. Eq. (39) can be expressed in matrix form as

$$\eta_i=\frac{|\underline{b}_{n_o}\underline{h}_i|^2}{\underline{h}_i^*\mathbf{A}_i\underline{h}_i},\quad i=1\to M \tag{40}$$

where

$$\mathbf{A}_i=\mathbf{F}_{i,0}^*\mathbf{F}_{i,0}+\sum\mathbf{F}_j^*\mathbf{F}_j>0 \tag{41}$$

represents a $K\times K$ positive-definite matrix, and $\mathbf{F}_i$ represents the $(N+K-1)\times K$ matrix

$$\mathbf{F}_i=\begin{pmatrix}x_i(1) & 0 & \cdots & 0 & 0\\ x_i(2) & x_i(1) & \ddots & \vdots & 0\\ \vdots & x_i(2) & \ddots & 0 & \vdots\\ x_i(N) & \vdots & \ddots & x_i(1) & 0\\ 0 & x_i(N) & \ddots & \vdots & x_i(1)\\ 0 & 0 & \ddots & x_i(N-1) & \vdots\\ \vdots & \vdots & \vdots & x_i(N) & x_i(N-1)\\ 0 & 0 & 0 & 0 & x_i(N)\end{pmatrix} \tag{42}$$

generated from the i^{th} input sequence $\underline{x}_i$. In (41), $\mathbf{F}_{i,0}$ represents the $(N+K-2)\times K$ matrix generated from $\mathbf{F}_i$ with its n_o row deleted. Finally in (40), $\underline{b}_{n_o}$ represents the row vector correspond to the n_o row of $\mathbf{F}_i$. Thus in general

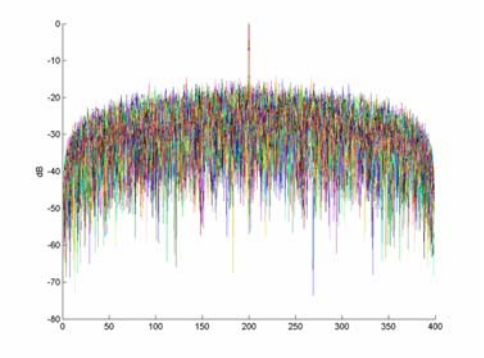
$$\underline{b}_{n_o}=\left[0,\ \cdots\ 0,\ x_i(N),\ x_i(N-1),\ \cdots\ x_i(1),\ 0,\ \cdots\ 0\right] \tag{43}$$

for $n_o>N$. The number of zeros in (43) depends on the value of n_o with respect to N. For example, with $n_o=N$ and $K=N$, the vector $\underline{b}_{n_o}$ is the same as the flipped version of $\underline{x}_i$ in (25). Schwarz inequality applied to (40) gives the optimum i^{th} receiver filter to be

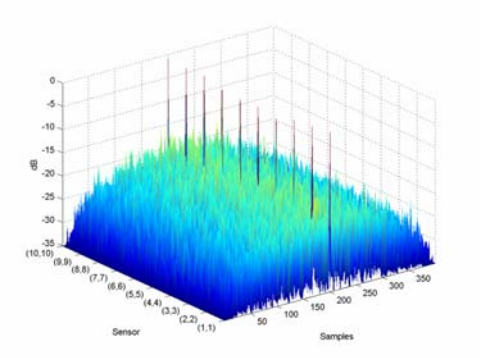
$$\underline{h}_i=\mathbf{A}_i^{-1}\underline{b}_{n_o}^*. \tag{44}$$

Observe that the receivers $\underline{h}_i$ in (44) are obtained non-iteratively and the freedom present at the input makes the unimodular constraint trivial to implement at the input. Although each receiver design depends on *all* input sequences through the matrix $\mathbf{A}_i$ in (41), nevertheless they can be determined in an uncoupled manner as in (44). Furthermore, the explicit design for the receiver allows freedom in deciding the output instant n_o at which the peak should be observed. This freedom allows the various peak outputs to be time aligned, a desired property in some radar applications.

Fig. 5 shows the receiver output using the new design approach in (44) for a ten channel randomly generated unimodular sequences of length 200. For comparison purposes, here the receiver length is set equal to 200. Notice that the auto/cross sidelobe terms are about 20 dB below the peak value, similar to that in Fig. 3. This represents performance similar to the CAN sequences obtained using (33) - (34). In this new approach, additional optimization can be carried out at the input to further reduce the sidelobe level and for compression in the Doppler domain.



(a) Front view



(b) Side view

Fig. 5 A ten channel unimodular sequence auto and cross terms at the output of the sidelobe minimizer receiver in (44). (a) Front view, (b) Side view.

VI. CONCLUSIONS

This paper reviews recent advances in designing unimodular sequences with good auto/cross correlation properties along with a new approach that emphasizes on independent receiver design by minimizing the sidelobes. Since the matched filter outputs represent auto and cross-correlations of the original sequences, the problem is to design constant modulus time-series whose auto correlations approximate the delta function, and the cross correlations approximate the all-zero function. In this context, CAN (cyclic-algorithm new) and WeCAN (Weighted CAN) have been proposed for designing such sequences with good auto-cross correlation properties. The equivalence of the CAN/WeCAN algorithms and the classic Gerchberg-Saxton (GS) algorithm involving the sequential magnitude substitution operations in the time and frequency domain is demonstrated here, with an interesting generalization of the GS algorithm to the multichannel case.

The design of unimodular sequences is further generalized here by considering the receiver design to be independent from the actual input sequences. The receiver design takes care of the sidelobe requirements and the freedom present at the input can be used to further minimize the output sidelobe levels.

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